Marouane's Method

AN IMPROVEMENT TO THE SECANT METHOD

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Abstract

This study presents an improvement to the secant method by reconstruction, in numerical analysis,the secant method is a root-finding algorithm that uses a succession of roots of secant lines to better approximate a root of a function F .The secant method can be thought of as a finite difference approximation of Newton's method. However the method was developed independently of Newton's method and predated the latter by over 3000 years.
Secant method which its convergence is superlinear is used in combination with bisection and inverse quadratic interpolation in Brent's and Zhang's method wich are one of the most powerful root finding algorithms.the new method presented in this study represents so much advantages in root finding algorithm for non-linear equations, compared to the secant method , this uses secant lines from 2 circles in each iteration, it then requires only one initial guess and its convergence is quadratic, this new method could replace the secant method in Brent's and Zhang's method to make the algorithm more quick and more efficient,some experimental tests presented in this study compares the performance of this new method to the secant method.
Keywords : Brent's method , Zhang Method , Secant method ,Improvement , Quadratic ,

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1. INTRODUCTION

.The secant method is defined by the recurrence relation

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

As can be seen from the recurrence relation the secant method requires two initial values x_0 and x_1 which should ideally be chosen to lie close to the root.The iterates x_n of the secant method converge to a root of f ,if the initial values x_0 and x_1 are sufficiently close to the root the order of convergence is α where

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Is the golden ratio. In particular ,the convergence is superlinear,but not quick quadratic [1]

.Zhang's method ,wich is proposed by Zhang (2011) [2]and corrected after that by Stage, Steven A. (2013) [3],,is an improvement to Brent's method [4]root-finding algorithm wich combine Bissection ,Secant and inverse quadratic interpolation,Zhang's improvements make the algorithm simpler and much more understandable, he shows one test example and find for that case that his method converges more rapidly than Brent's method.

2. IMPROVEMENTS TO THE SECANT METHOD

2.1Introduction to the new method

The parametric equation of a circle of a center (x_0, y_0) and radius r_0 is :

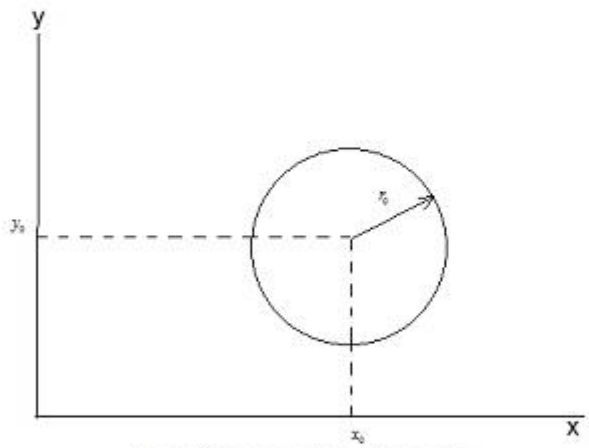
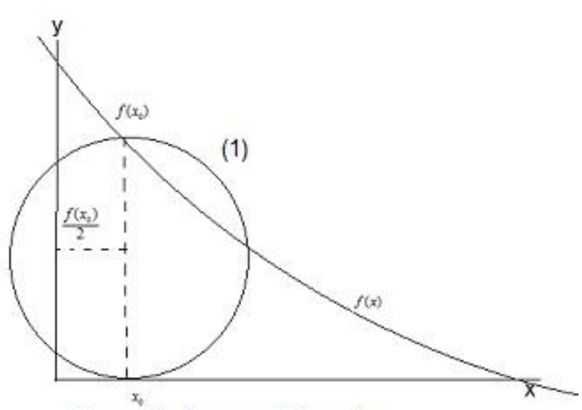


Figure (a) : representation of a cercle

$$\begin{cases} x = x_0 + r_0 \cos(\theta) \\ y = y_0 + r_0 \sin(\theta) \end{cases}$$

So the parametric equation of a circle (1) figure(b) tangent to the abscise axis in x_0 and where its diameter equal to

$[\![x_0, f(x_0)]\!]$is :

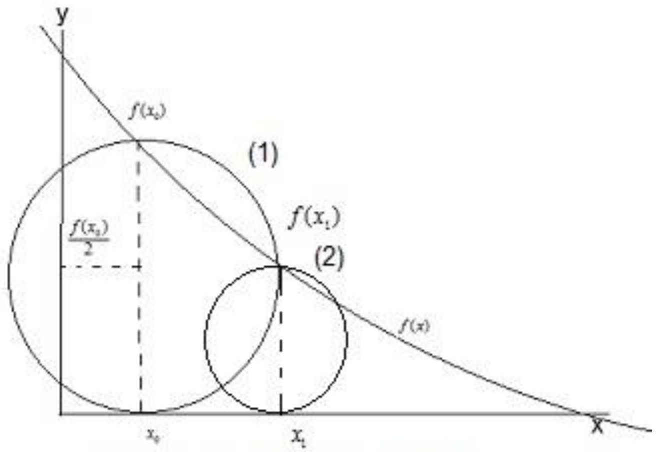


Figure(b): first step in iteration

Now a parametric equation of a circle (2) figure(c) tangent to the abscise axis in x_1 and which its coordinates depend on the first circle (1) is:

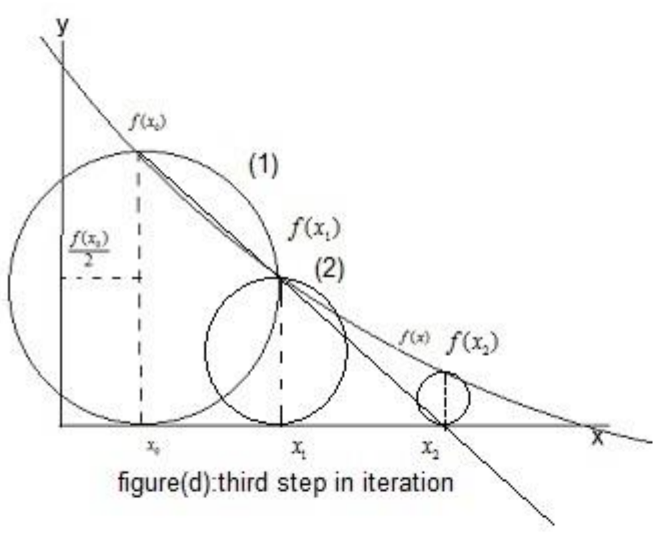
$$\begin{cases} x = x_1 + r_1 \cos(\theta_1) \\ y = y_1 + r_1 \sin(\theta_1) \end{cases} \quad (1)$$

When $(\theta_1 = \frac{\pi}{2})$ in equation (1) $x = x_1 = x_0 + \frac{|f(x_0)|}{2}$



figure(c):second step in iteration

.This is how we'll make our algorithm , for only one initial point x_0 ,we will ask our algorithm to plot fictively a first circle tangent to the abscise axis at x_0 ,its diameter is $[\![x_0, f(x_0)]\!]$ and its center is $[\![x = x_0, y = \frac{f(x_0)}{2}]\!]$ figure(b),the next circle will depend on the first one its diameter is $[\![x_1, f(x_1)]\!]$ its center is $[\![x_1 = x_0 + r_0, y = \frac{f(x_1)}{2}]\!]$ figure(c),then we construct a line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$,the intersection of this line with $y = 0$ will give us the next point x_2 figure(d),this way we have made one iteration of this new method , we will use this new value of x as x_2 and repeat the process until we reach a sufficiently high level of precision.



figure(d):third step in iteration

In figure (e) we can see the rapidity of this new method compared to secant method, we can conclude from this graphic comparison that the new method which requires only one initial guess x_0 makes two iterations in only one iteration ,then it converges more quickly than the secant method.

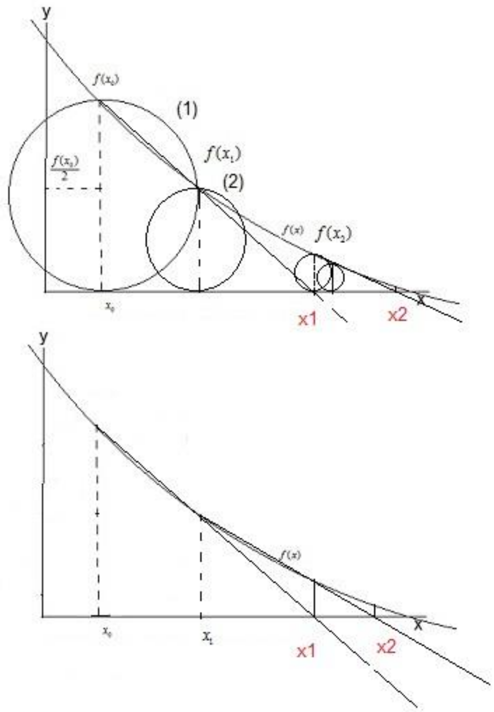


Figure (e) : graphic comparison between secant and the new method

2.2Developement of the method

The secant method is defined by:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

The modification we will give to this algorithm is very simple but it will have a good efficienty in convergence,moreover we will ne need just one initial value x_0 instead of two.

The line in figure(d) has the question:

$$y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1) + f(x_1) \qquad (2)$$

By replacing $x_1 = x_0 + \frac{|f(x_0)|}{2}$ and $f(x_1) = f(x_0 + \frac{|f(x_0)|}{2})$ and by rearranging the equation (2) for x and solving it when $y = 0$,equation(2) becomes :

$$x = x_0 + \frac{|f(x_0)|}{2} - \frac{|f(x_0)|}{2}\left(\frac{f(x_0 + \frac{|f(x_0)|}{2})}{f(x_0 + \frac{|f(x_0)|}{2}) - f(x_0)}\right) \qquad (3)$$

We conclude from equation (3) that this equation requires only one given initial value x_0 we then use this new value of x as x_2 and repeat the process to find the algorithm as follows:

$$x_2 = x_1 + \frac{|f(x_1)|}{2} - \frac{|f(x_1)|}{2}\left(\frac{f(x_1 + \frac{|f(x_1)|}{2})}{f(x_1 + \frac{|f(x_1)|}{2}) - f(x_1)}\right)$$

$$x_3 = x_2 + \frac{|f(x_2)|}{2} - \frac{|f(x_2)|}{2}\left(\frac{f(x_2 + \frac{|f(x_2)|}{2})}{f(x_2 + \frac{|f(x_2)|}{2}) - f(x_2)}\right)$$

$$\vdots$$

$$x_{n+1} = x_n + \frac{|f(x_n)|}{2} - \frac{|f(x_n)|}{2}\left(\frac{f(x_n + \frac{|f(x_n)|}{2})}{f(x_n + \frac{|f(x_n)|}{2}) - f(x_n)}\right) \qquad (4)$$

2.3Theorem:

Given a function f, let x be such that $f(x)=0$ and let and x_n be approximations to x. Assume r is a simple root $(f(r)=0, f'(r)\neq 0)$ and f is twice continuously differentiable , Marouane's method is defined by :

$$x_{n+1} = x_n + \frac{|f(x_n)|}{2} - \frac{|f(x_n)|}{2}\left(\frac{f(x_n + \frac{|f(x_n)}{2})}{f(x_n + \frac{|f(x_n)|}{2}) - f(x_n)}\right) \qquad n = 1, 2, \ldots$$

3. EXPERIMENTAL TESTS

Below some experimental tests for different functions $f(x)$ between the secant and the new method

3.1Example 1

Function tested $f(x_n) = x^2 - 2$ with initial values $x_0 = 0, x_1 = 3$ for secant method and $x_0 = 2$ for the new method, We will use 10 decimal digit arithmetic to find a solution and the resulting iteration is shown in Table 1.

	secant method			the new method					
n	x_n	$f(x_n)$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert}$	x_n	$f(x_n)$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert}$	$\frac{e_{n+1}}{e_n^{\ 2}}$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^3}$	$\frac{\log\lvert(x_{n+2})-(x_n+1)\rvert}{\log\lvert(x_{n+1})-(x_n)\rvert}$
1	0,6666666667	-1,5555555556	0,9035609149	1,6000000000	2,0000000000	0,1513295957	1,0057471264	5,8439475002	2,0336339887
2	1,0909090909	-0,8099173554	2,4118961320	1,4390804598	0,5600000000	0,0211393110	0,9404053449	35,6472078968	2,0380806133
3	1,5517241379	0,4078478002	2,2116338086	1,4147285708	0,0709525697	0,0004395540	0,8680768487	1658,6900075933	2,0208647907
4	1,3973902728	-0,0473004254	1,7287759487	1,4142137886	0,0014569290	0,0000001933	0,8538640168	3775716,8048628400	2,0102876087
5	1,4134291302	-0,0022180939	1,7172086905	1,4142135624	0,0000006400	0,0000000000	0,8543485614		
6	1,4142182573	0,0000132794	1,6675725778	1,4142135624	0,0000000000	0,0000000000			
7	1,4142135611	-0,0000000037	1,6492029942						
8	1,4142135624	0,0000000000							
9	1,4142135624	0,0000000000							
10	1,4142135624	0,0000000000							

TABLE 1: Comparisons between the results from the two methods.

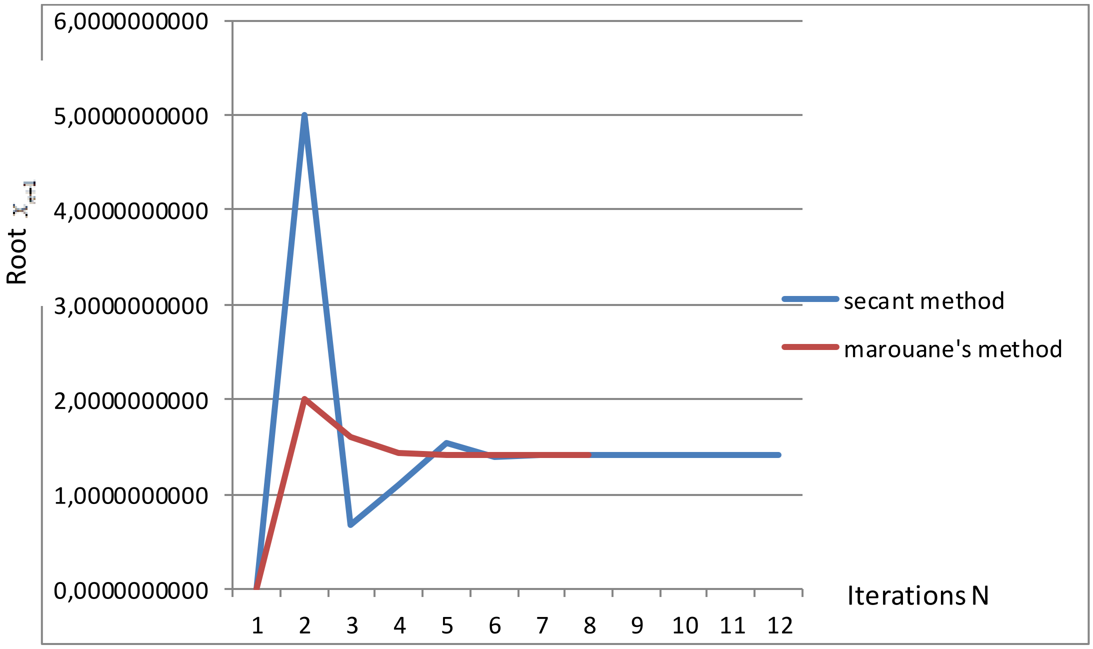


Figure 1: Graphic convergence comparison.

Table 1 was done through Microsoft Excel,as we see the new method can reach the real root more quickly with fewer iterations steps,the last column of the new method shows the speed of convergence $\frac{\log|(x_{n+2})-(x_n+1)|}{\log|(x_{n+1})-(x_n)|}\approx 2$,Figure1 shows a graphic comparison , we can see how fast is the new method to approach the root compared to the secant method, some other comparison examples are shown below

3.2Example 2

Function tested $f(x)=x^3+x^2+x+1$ with initial values $x_0=0, x_1=1$ for secant method and $x_0=0$ for the new method

	secant method			the new method					
n	x_n	$f(x_n)$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert}$	x_n	$f(x_n)$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert}$	$\frac{e_{n+1}}{e_n^{\ 2}}$	$\frac{\lvert e_{n+1}\rvert}{\lvert e_n\rvert^3}$	$\frac{\log\lvert(x_{n+2})-(x_n+1)\rvert}{\log\lvert(x_{n+1})-(x_n)\rvert}$
1	-0,3333333333	0,7407407407	14,9717860894	-0,5714285714	0,5685131195	0,5259315573	2,4367823185	0,7765410321	3,2903167950
2	-1,2857142857	-0,7580174927	2,6547606531	-1,3671125938	-1,0532448731	0,1359891140	0,6609804175	30,0898521208	1,8123480169
3	-0,8040345821	0,3226514969	1,4202053715	-0,9486372568	0,0975847253	0,0974469111	0,3249632701	0,9999158226	3,0000162050
4	-0,9478479782	0,0990062217	1,6021233392	-1,0055453471	-0,0111523665	0,0000307502	1,7123560216	11732009,1139789000	1,9555034363
5	-1,0115131773	-0,0232929871	1,6792508249	-0,9999998295	0,0000003411	0,0000003412	0,0055450508		
6	-0,9993875660	0,0012241181	1,6015994290	-1,0000000000	0,0000000000				
7	-0,9999929874	0,0000140251	1,6234303251	-1,0000000000	0,0000000000				
8	-1,0000000043	-0,0000000086	1,6160803060						
9	-1,0000000000	0,0000000000	#NOMBRE!						
10	-1,0000000000	0,0000000000	#DIV/0!						
11	-1,0000000000	0,0000000000	#DIV/0!						

TABLE 2: Comparisons between the results from the two methods for the example 2

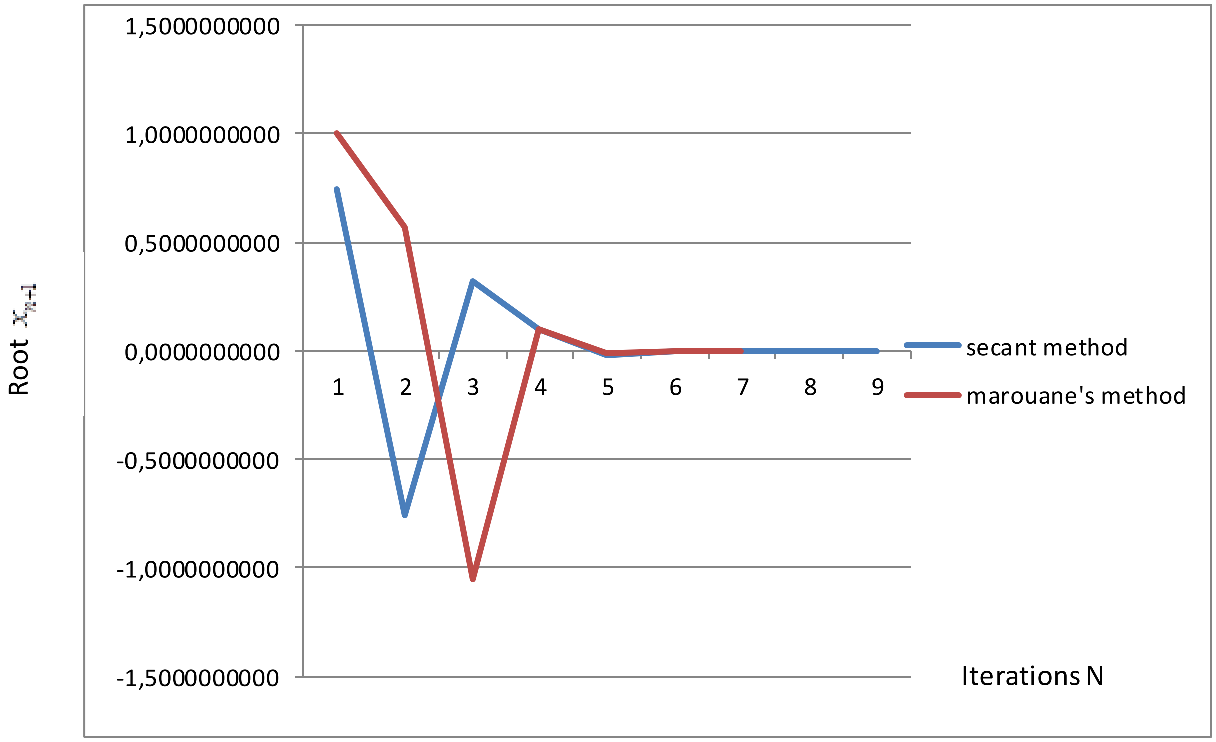


Figure 2: Graphic convergence comparison for example 2

3.3Example 3

Function tested $\cos(x)-x^3$ with initial value $x_0=-2, x_1=0$ for secant method and $x_0=-2$ for the new method

	secant method			the new method					
n	x_n	$f(x_n)$	$\frac{\vert e_{n+1}\vert}{\vert e_n\vert}$	x_n	$f(x_n)$	$\frac{\vert e_{n+1}\vert}{\vert e_n\vert}$	$\frac{e_{n+1}}{e_n^{\ 2}}$	$\frac{\vert e_{n+1}\vert}{\vert e_n\vert^3}$	$\frac{\log\vert(x_{n+2})-(x_n+1)\vert}{\log\vert(x_{n+1})-(x_n)\vert}$
1	0,3037734819	0,9622235067	0,9902113371	0,1212126390	0,9908818211	0,1519249192	0,1267434709	25,1559386661	1,1063918607
2	4,1152150220	-0,9220253692	-2,3033988366	1,3198930772	-2,0511301103	0,8342676263	4,5811336193	26,0689816107	1,2695631522
3	0,3533676074	0,9525642064	-0,0243925868	1,1377836485	-1,0533123496	0,6017272949	3,9606082596	35,3689046272	1,5094537462
4	0,4006415257	0,9439711887	-2,2442962829	0,9858556476	-0,4060143764	0,2955947253	3,2333978487	97,8155635150	1,7308223003
5	1,4779235361	-0,1784849025	12,3416723007	0,8944364227	-0,0896054839	0,0714292996	2,6432739529	1266,9887642923	1,8569168553
6	0,6317815789	0,9288102436	0,9618897277	0,8674133820	-0,0058455967	0,0047205649	2,4455884256	266808,5076938000	1,9234575414
7	0,7589935935	0,9486805683	1,6973374296	0,8654831451	-0,0000274141	0,0000221518	2,4311080286	#DIV/0!	#DIV/0!
8	0,8966051182	0,9845828091	1,6988257514	0,8654740333	-0,0000000006	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
9	0,8620902226	0,9755942313	1,6092815883	0,8654740331	0,0000000000	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
10	0,8653736537	0,9764788133	1,6214400916		#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
11	0,8654743636	0,9765058808	1,6189467572		#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
12	0,8654740331	0,9765057920	#NOMBRE!		#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
13	0,8654740331	0,9765057920	#DIV/0!		#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!
14	0,8654740331	0,9765057920	#DIV/0!		#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!	#DIV/0!

TABLE 3: Comparisons between the results from the two methods for the example 3

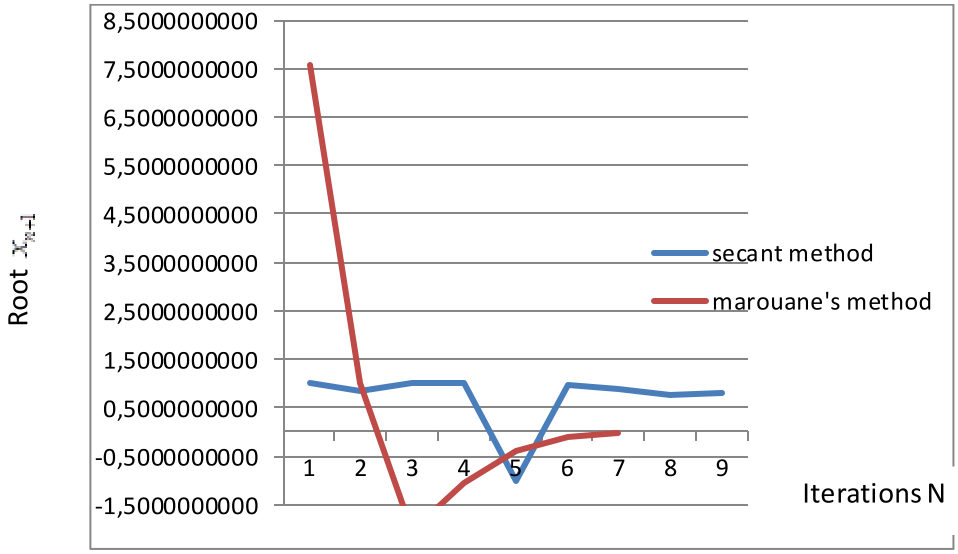


Figure 3: Graphic convergence comparison for example 3

4. CONCLUSIONS

This study proposes an improvement to the secant method and a comparative experiments tests were conducted.Experimental tests indicated that the proposed method converges faster in quadratic order with only one initial guess,in short,the proposed method shows lot of advantadges compared to the secant method and could be implemented in Brent's [4] and corrected Zhang's method [3] .

5.REFERENCES:

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